

# A closed-form representation of mean-variance hedging for additive processes via Malliavin calculus

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## Abstract

We derive an explicit closed-form representation of mean-variance hedging strategies for models whose asset price follows an exponential additive process. Our representation is given in terms of Malliavin calculus for Lévy processes. In addition, we develop an approximation method to compute mean-variance hedging strategies for exponential Lévy models, and illustrate numerical results.

**Keywords:** Mean-variance hedging, Additive processes, Malliavin calculus, Fast Fourier transform.

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## 1 Introduction

Hedging problem for contingent claims in incomplete markets is a centerpiece of mathematical finance. Actually, many hedging methods for incomplete markets have been suggested. Above all, we focus on mean-variance hedging (MVH), which has been studied very well for about three decades. Our aim of this paper is twofold: The first is to derive an explicit closed-form representation of MVH strategies for exponential additive models using Malliavin calculus for Lévy processes. The second is to develop a simple numerical method for exponential Lévy models, and to illustrate numerical results.

We consider throughout an incomplete financial market in which only one risky asset and one riskless asset are tradable. Let  $T > 0$  be the maturity of our market, and suppose that the interest rate of the riskless asset is 0 for sake of

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simplicity. The risky asset price process, denoted by  $S$ , is given as a solution to the following stochastic differential equation:

$$dS_t = S_{t-} \left[ \alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z} \tilde{N}(dt, dz) \right], \quad S_0 > 0,$$

where  $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ ,  $W$  is a one-dimensional standard Brownian motion, and  $\tilde{N}$  is the compensated version of a homogeneous Poisson random measure  $N$ . Here,  $\alpha$  and  $\beta$  are deterministic measurable functions on  $[0, T]$ , and  $\gamma$  is also deterministic and jointly measurable on  $[0, T] \times \mathbb{R}_0$ . We assume  $\gamma > -1$ , which ensures the positivity of  $S$ . Then,  $S$  is given as an exponential of an additive process, that is,  $\log(S)$  is continuous in probability and has independent increments. In addition, when  $\alpha$  and  $\beta$  are given by a real number and a non-negative real number, respectively, and  $\gamma_{t,z} = e^z - 1$ , we call  $S$  an exponential Lévy process. Let  $H$  be a square integrable random variable. We consider its value as the payoff of a contingent claim at the maturity  $T$ . In principle, since our market is incomplete, we cannot find a replicating strategy for  $H$ , that is, there is no pair  $(c, \vartheta) \in \mathbb{R} \times \Theta$  satisfying

$$H = c + G_T(\vartheta),$$

where  $\Theta$  is a set of predictable processes, which is considered as the set of all admissible strategies in some sense, and  $G(\vartheta)$  denotes the gain process induced by  $\vartheta$ , that is,  $G(\vartheta) := \int_0^\cdot \vartheta_u dS_u$ . Note that each pair  $(c, \vartheta) \in \mathbb{R} \times \Theta$  represents a self-financing strategy. Instead of finding the replicating strategy, we consider the following minimization problem:

$$\min_{c \in \mathbb{R}, \vartheta \in \Theta} \mathbb{E} \left[ (H - c - G_T(\vartheta))^2 \right],$$

and call its solution  $(\hat{c}^H, \hat{\vartheta}^H) \in \mathbb{R} \times \Theta$  the MVH strategy for claim  $H$  if it exists. In other words, the MVH strategy defined as the self-financing one minimizing the corresponding  $L^2$ -hedging error over  $\mathbb{R} \times \Theta$ . Remark that  $\hat{c}^H$  gives the initial cost which is regarded as the corresponding price of  $H$ , and  $\hat{\vartheta}_t^H$  represents the number of shares of the risky asset in the strategy at time  $t$ .

In addition to MVH strategy, locally risk-minimizing (LRM) strategy has been studied well as alternative hedging method in quadratic way. Being different from the MVH approach, an LRM strategy is given as a replicating strategy which is not necessarily self-financing. Thus, we need to take an additional cost process into account. Roughly speaking, an LRM strategy is defined as the one minimizing in the  $L^2$ -sense the risk caused by such an additional cost process among all replicating strategies which are not necessarily self-financing. For more details, see Schweizer [17] and [18]. As for expressions of LRM strategies, Arai and Suzuki [4] obtained an explicit form for Lévy markets using Malliavin calculus for Lévy processes. Here, Lévy market is a similar model framework to ours, but the coefficient functions  $\alpha$ ,  $\beta$  and  $\gamma$  may have randomness. In other words, our model is a Lévy market with deterministic coefficients.

There is much literature on MVH strategies for jump type models. Among others, Arai [1], Černý and Kallsen [6], and Jeanblanc et al. [13] provided feedback-form representations of MVH strategies for general model frameworks using semimartingale approaches, duality approaches or backward stochastic differential equations, but their representations are not given concretely for concrete models. Here, a representation of the MVH strategy  $\tilde{\vartheta}_t^H$  is said to be feedback-form if it includes  $G_{t-}(\tilde{\vartheta}^H)$ . Moreover, Lim [15] considered a Lévy market and gave a closed-form expression of  $\tilde{\vartheta}^H$ , that is, an expression without using the values of  $\tilde{\vartheta}^H$  up to time  $t$ . However, he restricted  $H$  to be bounded, and his expression is not an explicit form, since it includes solutions to backward stochastic differential equations. On the other hand, as researches on explicit representations for concrete models, Hubalek et al. [11] obtained a representation in feedback-form for exponential Lévy models, and also their results have been extended to the additive process case and affine stochastic volatility models by Goutte et al. [9], and Kallsen and Vierthauer [14], respectively. The discussion in [11] is based on bilateral Laplace transforms and the Föllmer-Schweizer (FS) decomposition, which is an expression of  $H$  by the sum of a stochastic integral with respect to  $S$  and a residual martingale. In addition, combining their Theorems 3.1 and 3.3, they also gave an explicit closed-form representation.

As mentioned before Theorem 3.3 in [11], an explicit closed-form representation might be preferred to one in feedback-form from a numeric analytical point of view, since  $G_{t-}(\tilde{\vartheta}^H)$  is approximated with an involved recursive calculation, which is very time-consuming and entails a drop of the accuracy. For more details on this matter, see Remark 4.1. In addition, the closed-form representation obtained in [11] is given as a direct extension of the feedback-form one using a general stochastic exponential, namely, it is also not appropriate for the development of numerical schemes. Besides, their closed-form representation includes a stochastic integral with respect to the quadratic variation of  $S$  which is not observable. Therefore, we shall derive a different explicit closed-form representation for exponential additive models in order to develop a simple numerical method for which we need discrete observational data of  $S$  alone. This is the first main purpose of this paper. In particular, we make use of results of [4], and represent  $\tilde{\vartheta}^H$  in terms of Malliavin calculus for Lévy processes. Furthermore, we rely on the argument of [1], which is based on a different decomposition of  $H$  from the FS decomposition. As one more advantage of our representation, path-dependent options are covered as seen in Examples 3.5 and 3.6, while [11] excluded them.

As the second goal of this paper, we shall develop a numerical method to compute  $\tilde{\vartheta}_t^H$  for non-zero given  $t \in (0, T]$  for exponential Lévy models using the closed-form representation of  $\tilde{\vartheta}^H$  obtained in this paper and the fast Fourier transforms (FFT). There is some literature on numerical analysis of MVH strategies, say De Franco et al. [8], [9], [11] and so on, all of which computed the initial hedge  $\tilde{\vartheta}_0^H$  or the hedging error induced by MVH strate-

gies or both alone, but no one has developed any numerical scheme of  $\tilde{\vartheta}_t^H$  for non-zero given  $t \in (0, T]$  so far as we know. The most difficulty on the development of numerical methods for MVH strategies lies in the fact that  $\tilde{\vartheta}_t^H$  for given  $t \in (0, T]$  is depending on the whole trajectory of  $S$  up to time  $t$ . From a practical point of view, we cannot observe the trajectory of  $S$  continuously, however. Thus, we compute  $\tilde{\vartheta}^H$  approximately using discrete observational data of  $S$  in this paper. The numerical method developed in this paper has the following two features. First, our method is based on the explicit closed-form representation obtained in this paper. As a result, we can avoid to use any involved recursive calculation, although a simple recursive calculation is needed in order to compute a discretization of a stochastic exponential. In addition, we use observational data of  $S$  alone, in other words, our method do not need any unobservable data. Second, we make use of results of Arai et al. [3], which developed an FFT-based numerical scheme of LRM strategies for exponential Lévy models using results of [4]. In principle, computing an MVH strategy is time-consuming since its expression includes stochastic integrals. Nevertheless, an FFT-based approach enables us to achieve high speed computation. Moreover, results of [3] is also used to compute the LRM strategy, which is included in our representation of  $\tilde{\vartheta}^H$ .

An outline of this paper is as follows: Model description and mathematical preliminaries are given in Section 2. In particular, we introduce the variance-optimal martingale measure and Malliavin calculus for Lévy processes. The main result and its proof are addressed in Section 3. Subsection 3.2 introduces some examples. Section 4 is devoted to develop numerical scheme and to introduce numerical results.

## 2 Preliminaries

### 2.1 Model description

We consider throughout a financial market being composed of one risky asset and one riskless asset with finite time horizon  $T > 0$ . For simplicity, we assume that the interest rate of the market is given by 0, that is, the price of the riskless asset is 1 at all times.

Let  $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$  be a one-dimensional Wiener space on  $[0, T]$ ; and  $W$  its coordinate mapping process, that is, a one-dimensional standard Brownian motion with  $W_0 = 0$ .  $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$  denotes the canonical Lévy space for a pure jump Lévy process  $J$  on  $[0, T]$  with Lévy measure  $\nu$ , that is,  $\Omega_J = \cup_{n=0}^{\infty} ([0, T] \times \mathbb{R}_0)^n$ , where  $J_t(\omega_J) = \sum_{i=1}^n z_i \mathbf{1}_{\{t_i \leq t\}}$  for  $t \in [0, T]$  and  $\omega_J = ((t_1, z_1), \dots, (t_n, z_n)) \in ([0, T] \times \mathbb{R}_0)^n$ . Note that  $([0, T] \times \mathbb{R}_0)^0$  represents an empty sequence. For more details on the canonical Lévy space, see Solé et al. [19]. Now, we assume that  $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$ ; and denote

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J).$$

Let  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  be the canonical filtration completed for  $\mathbb{P}$ . Let  $X$  be a square integrable centered Lévy process on  $(\Omega, \mathcal{F}, \mathbb{P})$  represented as

$$X_t = W_t + J_t - t \int_{\mathbb{R}_0} z \nu(dz).$$

Denoting by  $N$  the Poisson random measure defined as  $N(t, A) := \sum_{s \leq t} \mathbf{1}_A(\Delta X_s)$ ,  $A \in \mathcal{B}(\mathbb{R}_0)$  and  $t \in [0, T]$ , where  $\Delta X_s := X_s - X_{s-}$ , we have  $J_t = \int_0^t \int_{\mathbb{R}_0} z N(ds, dz)$ .

In addition, we define its compensated measure as  $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ . Thus,  $X$  is expressed as

$$X_t = W_t + \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(du, dz). \quad (2.1)$$

The fluctuation of the risky asset is assumed to be given by a solution to the following stochastic differential equation:

$$dS_t = S_{t-} \left[ \alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z} \tilde{N}(dt, dz) \right], \quad S_0 > 0, \quad (2.2)$$

where  $\alpha$  and  $\beta$  are deterministic measurable functions on  $[0, T]$ , and  $\gamma$  is a deterministic jointly measurable function on  $[0, T] \times \mathbb{R}_0$ . In addition, we denote

$$\Gamma_t := \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz), \quad \text{and} \quad \lambda_t := \frac{\alpha_t}{S_{t-}(\beta_t^2 + \Gamma_t)}$$

for  $t \in [0, T]$ . Now, we assume throughout this paper the following:

**Assumption 2.1** 1.  $\gamma_{t,z} > -1$  for any  $(t, z) \in [0, T] \times \mathbb{R}_0$ .

2.  $\sup_{t \in [0, T]} (|\alpha_t| + \beta_t^2 + \Gamma_t) < C$  for some  $C > 0$ .

3. There exists an  $\varepsilon > 0$  such that

$$\lambda_t S_{t-} \gamma_{t,z} < 1 - \varepsilon \quad \text{and} \quad \beta_t^2 + \Gamma_t > \varepsilon, \quad (t, z, \omega)\text{-a.e.}$$

**Remark 2.2** 1. Under Assumption 2.1, (2.2) has a solution  $S$  satisfying the so-called structure condition (SC), that is,  $S$  has the following three properties:

(a)  $S$  is a semimartingale of the space  $\mathcal{S}^2$ , that is, a special semimartingale with the canonical decomposition  $S = S_0 + M + A$  such that

$$\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})} < \infty, \quad (2.3)$$

where  $dM_t = S_{t-}(\beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z} \tilde{N}(dt, dz))$  and  $dA_t = S_{t-} \alpha_t dt$ .

(b) We have  $A = \int \lambda d\langle M \rangle$ .

(c) The mean-variance trade-off process  $K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s$  is finite, that is,  $K_T$  is finite  $\mathbb{P}$ -a.s.

The SC is closely related to the no-arbitrage condition. For more details on the SC, see [17] and [18].

2. The process  $K$  as well as  $A$  is continuous. In particular,  $K$  is deterministic.
3. (2.3) implies that  $\sup_{t \in [0, T]} |S_t| \in L^2(\mathbb{P})$  by Theorem V.2 of Protter [16].
4. Item 1 in Assumption 2.1 ensures the positivity of  $S$ . Thus, under Assumption 2.1,  $S$  is an exponential of an additive process, that is, its logarithm  $\log(S)$  has the following properties:
  - (a) (Continuous in probability):  $\log(S)$  has no fixed time of discontinuity.
  - (b) (Independent increments):  $\log(S_{t_1}) - \log(S_{t_2})$  is independent of  $\mathcal{F}_{t_2}$  for  $0 \leq t_2 < t_1 \leq T$ .

## 2.2 Variance-optimal martingale measure

In this subsection, we discuss the variance-optimal martingale measure, which is indispensable to discuss MVH strategies. Roughly speaking, the variance-optimal martingale measure is defined as an equivalent martingale measure whose density minimizes its  $L^2(\mathbb{P})$ -norm.

We start with the definition of admissible strategies. We denote by  $\Theta$  the space of all  $\mathbb{R}$ -valued predictable  $S$ -integrable processes  $\vartheta$  whose stochastic integral  $\int_0^t \vartheta_u dS_u$  is a semimartingale of the space  $\mathcal{S}^2$ . Throughout this paper, we regard  $\Theta$  as the set of all admissible strategies. Remark that Assumption 2.1 implies

$$\Theta = \left\{ \vartheta \text{ } \mathbb{R}\text{-valued predictable } S\text{-integrable process} : \mathbb{E} \left[ \int_0^T \vartheta_u^2 S_{u-}^2 du \right] < \infty \right\}$$

Next, we define the variance-optimal martingale measure as follows:

**Definition 2.3** 1. A signed measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{F})$  is called a signed  $\Theta$ -martingale measure if  $\mathbb{Q}(\Omega) = 1$ ,  $\mathbb{Q} \ll \mathbb{P}$ ,  $d\mathbb{Q}/d\mathbb{P} \in L^2(\mathbb{P})$  and

$$\mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \cdot G_T(\vartheta) \right] = 0$$

for any  $\vartheta \in \Theta$ , where  $G_T(\vartheta) = \int_0^T \vartheta_u dS_u$ . We denote by  $\mathcal{P}_s(\Theta)$  the set of all signed  $\Theta$ -martingale measures, and  $\mathcal{D}_s(\Theta) := \{d\mathbb{Q}/d\mathbb{P} | \mathbb{Q} \in \mathcal{P}_s(\Theta)\}$ .

2.  $\mathbb{Q} \in \mathcal{P}_s(\Theta)$  is called an *equivalent martingale measure* if it is a probability measure equivalent to  $\mathbb{P}$ .
3. An equivalent martingale measure  $\mathbb{P}^* \in \mathcal{P}_s(\Theta)$  is called the *variance-optimal martingale measure* if its density  $d\mathbb{P}^*/d\mathbb{P}$  minimizes  $\|D\|_{L^2(\mathbb{P})}$  over all  $D \in \mathcal{D}_s(\Theta)$ .

We shall show that the variance-optimal martingale measure is given by a probability measure  $\mathbb{P}^*$  defined as

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}_T \left( - \int_0^\cdot \lambda_u dM_u \right)$$

where  $\mathcal{E}(Y)$  represents the stochastic exponential of  $Y$ , that is, a solution to the stochastic differential equation  $d\mathcal{E}_t(Y) = \mathcal{E}_{t-}(Y)dX_t$  with  $\mathcal{E}_0(Y) = 1$ . Henceforth, we denote  $D^* := \frac{d\mathbb{P}^*}{d\mathbb{P}}$  and  $Z_t := \mathcal{E}_t(-\int_0^\cdot \lambda_u dM_u)$ . Note that  $Z$  is a positive square integrable martingale under Assumption 2.1 as seen in Example 2.8 of [4], that is,  $\mathbb{P}^*$  is an equivalent martingale measure. We prove the following proposition in order to make sure that our setting satisfies all the standing assumptions in [1].

**Proposition 2.4** *Under Assumption 2.1, we have the following:*

1.  $\mathbb{P}^*$  is the variance-optimal martingale measure.
2.  $Z$  satisfies the reverse Hölder inequality  $R_2(\mathbb{P})$ , that is, there is a constant  $C > 0$  such that, for any stopping time  $\tau \leq T$ , we have

$$\mathbb{E} \left[ \left( \frac{Z_T}{Z_\tau} \right)^2 \middle| \mathcal{F}_\tau \right] \leq C.$$

3. There is a  $C > 0$  such that  $Z_{t-} \leq CZ_t$  for any  $t \in [0, T]$ .

*Proof.* To see item 1, we define  $\tilde{Z}_t := \mathbb{E}_{\mathbb{P}^*}[D^* | \mathcal{F}_t]$ . Note that  $\tilde{Z}_T = D^*$  and  $\tilde{Z}_0 = \mathbb{E}[(D^*)^2]$ . Now, we calculate  $\tilde{Z}$  as follows:

$$\begin{aligned} \tilde{Z}_t &= \mathbb{E}_{\mathbb{P}^*} \left[ \mathcal{E}_T \left( - \int_0^\cdot \lambda_u dM_u \right) \middle| \mathcal{F}_t \right] \\ &= \exp \left( \int_0^T \lambda_u dA_u \right) \mathbb{E}_{\mathbb{P}^*} \left[ \mathcal{E}_T \left( - \int_0^\cdot \lambda_u dS_u \right) \middle| \mathcal{F}_t \right] \\ &= \exp \left( \int_0^T \lambda_u dA_u \right) \mathcal{E}_t \left( - \int_0^\cdot \lambda_u dS_u \right) = \mathbb{E}[(D^*)^2] \mathcal{E}_t \left( - \int_0^\cdot \lambda_u dS_u \right) \end{aligned}$$

for any  $t \in [0, T]$ , since  $\int_0^\cdot \lambda_u dA_u$  is deterministic by Remark 2.2. Note that  $\tilde{Z}$  is a solution to the following equation:

$$\tilde{Z}_t = \tilde{Z}_0 - \int_0^t \tilde{Z}_{u-} \lambda_u dS_u.$$

For any  $Q \in \mathcal{P}_s(\Theta)$ , we have

$$\begin{aligned} \left\| \frac{dQ}{d\mathbb{P}} \right\|_{L^2(\mathbb{P})}^2 &= \left\| \frac{dQ}{d\mathbb{P}} - D^* \right\|_{L^2(\mathbb{P})}^2 + 2\mathbb{E} \left[ \frac{dQ}{d\mathbb{P}} D^* \right] - \|D^*\|_{L^2(\mathbb{P})}^2 \\ &\geq 2\tilde{Z}_0 \mathbb{E} \left[ \frac{dQ}{d\mathbb{P}} \mathcal{E}_T \left( - \int_0^\cdot \lambda_u dS_u \right) \right] - \tilde{Z}_0 = \tilde{Z}_0, \end{aligned}$$

from which item 1 follows.

Item 2 holds true from Proposition 3.7 of Choulli et al. [7]. In addition, since we can find an  $\varepsilon > 0$  such that  $\lambda_t \Delta M_t < 1 - \varepsilon$  for any  $t \in [0, T]$  from the view of item 3 of Assumption 2.1, we have  $Z_t / Z_{t-} = 1 - \lambda_t \Delta M_t > 1 - (1 - \varepsilon) = \varepsilon$ , from which item 3 follows.  $\square$

**Remark 2.5** *The essence of the above proof lies in the fact that  $\int_0^T \lambda_u dA_u (= K_T)$  is deterministic. General speaking, when  $K_T$  is deterministic, the variance-optimal martingale measure becomes as well the minimal martingale measure, which is defined as an equivalent martingale measure under which any square-integrable  $\mathbb{P}$ -martingale orthogonal to  $M$  remains a martingale. Actually, Example 2.8 of [4] showed that  $\mathbb{P}^*$  is the minimal martingale measure. Note that the minimal martingale measure is essential in the LRM approach.*

Now, we prepare some notation for later use. The Girsanov theorem implies that

$$W_t^{\mathbb{P}^*} := W_t + \int_0^t \lambda_u S_{u-} \beta_u du$$

is a one-dimensional standard Brownian motion under  $\mathbb{P}^*$ . Moreover, denoting

$$\nu_t^{\mathbb{P}^*}(dz) := (1 - \lambda_t S_{t-} \gamma_{t,z}) \nu(dz), \quad (2.4)$$

and  $\tilde{N}^{\mathbb{P}^*}(dt, dz) := N(dt, dz) - \nu_t^{\mathbb{P}^*}(dz)dt$  for  $t \in [0, T]$  and  $z \in \mathbb{R}_0$ , we have

$$\mathbb{E}_{\mathbb{P}^*} \left[ \tilde{N}^{\mathbb{P}^*}(A, B) \right] = 0$$

for any  $A \in \mathcal{B}([0, T])$  and  $B \in \mathcal{B}(\mathbb{R}_0)$ .

### 2.3 Malliavin calculus

One of our aims in this paper is to obtain a closed-form representation of MVH strategies in terms of Malliavin calculus for Lévy processes. Now, we prepare some notation and definitions with respect to Malliavin calculus. We adapt the canonical Lévy space framework undertaken by [19], which is a Malliavin calculus on the Lévy process  $X$  given in (2.1). First of all, we define measures  $q$  and  $Q$  on  $[0, T] \times \mathbb{R}$  as

$$q(E) := \int_E \delta_0(dz)dt + \int_E z^2 \nu(dz)dt,$$



and

$$Q(E) := \int_E \delta_0(dz) dW_t + \int_E z \tilde{N}(dt, dz),$$

where  $E \in \mathcal{B}([0, T] \times \mathbb{R})$  and  $\delta_0$  is the Dirac measure at 0. For  $n \in \mathbb{N}$ , we denote by  $L_{T,q,n}^2$  the set of product measurable, deterministic functions  $f : ([0, T] \times \mathbb{R})^n \rightarrow \mathbb{R}$  satisfying

$$\|h\|_{L_{T,q,n}^2}^2 := \int_{([0,T] \times \mathbb{R})^n} |f((t_1, z_1), \dots, (t_n, z_n))|^2 q(dt_1, dz_1) \cdots q(dt_n, dz_n) < \infty.$$

For  $n \in \mathbb{N}$  and  $f \in L_{T,q,n}^2$ , we define

$$I_n(f) := \int_{([0,T] \times \mathbb{R})^n} f((t_1, z_1), \dots, (t_n, z_n)) Q(dt_1, dz_1) \cdots Q(dt_n, dz_n).$$

Formally, we denote  $L_{T,q,0}^2 := \mathbb{R}$  and  $I_0(f) := f$  for  $f \in \mathbb{R}$ . Under this setting, any  $F \in L^2(\mathbb{P})$  has the unique representation  $F = \sum_{n=0}^{\infty} I_n(f_n)$  with functions  $f_n \in L_{T,q,n}^2$  that are symmetric in the  $n$  pairs  $(t_i, z_i)$ ,  $1 \leq i \leq n$ , and we have  $\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|_{L_{T,q,n}^2}^2$ . Now, we define a Malliavin derivative operator  $D$  as follows:

**Definition 2.6** 1. Let  $\mathbb{D}^{1,2}$  denote the set of random variables  $F \in L^2(\mathbb{P})$  with  $F = \sum_{n=0}^{\infty} I_n(f_n)$  satisfying  $\sum_{n=1}^{\infty} n! \|f_n\|_{L_{T,q,n}^2}^2 < \infty$ .

2. For any  $F \in \mathbb{D}^{1,2}$ ,  $DF : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  is defined by

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t, z), \cdot))$$

for  $q$ -a.e.  $(t, z) \in [0, T] \times \mathbb{R}$ ,  $\mathbb{P}$ -a.s.

Let  $H \in L^2(\mathbb{P})$  be a random variable representing the payoff of a claim to hedge. In addition to Assumption 2.1, we assume throughout the following:

**Assumption 2.7** 1.  $Z_T H \in L^2(\mathbb{P})$ .

2.  $H \in \mathbb{D}^{1,2}$  and  $Z_T D_{t,z}H + H D_{t,z}Z_T + z D_{t,z}H \cdot D_{t,z}Z_T \in L^2(q \times \mathbb{P})$ .

Under Assumptions 2.1 and 2.7, Example 3.9 of [4] implies that  $H$  is described as

$$H = \mathbb{E}_{\mathbb{P}^*}[H] + \int_0^T I_t dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} J_{t,z} \tilde{N}^{\mathbb{P}^*}(dt, dz), \quad (2.5)$$

where  $I_t := \mathbb{E}_{\mathbb{P}^*}[D_{t,0}H | \mathcal{F}_{t-}]$ , and  $J_{t,z} := \mathbb{E}_{\mathbb{P}^*}[z D_{t,z}H | \mathcal{F}_{t-}]$  for  $t \in [0, T]$  and  $z \in \mathbb{R}_0$ . Now, we denote additionally for later use  $K_t := \int_{\mathbb{R}_0} J_{t,z} \gamma_{t,z} \nu(dz)$  and

$$H_t := \mathbb{E}_{\mathbb{P}^*}[H | \mathcal{F}_t] = \mathbb{E}_{\mathbb{P}^*}[H] + \int_0^t I_u dW_u^{\mathbb{P}^*} + \int_0^t \int_{\mathbb{R}_0} J_{u,z} \tilde{N}^{\mathbb{P}^*}(du, dz)$$

for  $t \in [0, T]$ .

### 3 Main results

We derive in this section a closed-form expression of the MVH strategy for claim  $H \in L^2(\mathbb{P})$  in terms of Malliavin calculus. As mentioned in Introduction, the MVH strategy for  $H$  is defined as a pair  $(\tilde{c}^H, \tilde{\vartheta}^H) \in \mathbb{R} \times \Theta$  which minimizes

$$\min_{c \in \mathbb{R}, \vartheta \in \Theta} \mathbb{E} \left[ (H - c - G_T(\vartheta))^2 \right]. \quad (3.1)$$

The following theorem is shown in Subsection 3.1, and some examples will be introduced in Subsection 3.2.

**Theorem 3.1** *Under Assumptions 2.1 and 2.7, the MVH strategy  $(\tilde{c}^H, \tilde{\vartheta}^H) \in \mathbb{R} \times \Theta$  for claim  $H \in L^2(\mathbb{P})$  is represented in closed-form as*

$$\tilde{c}^H = \mathbb{E}_{\mathbb{P}^*}[H]$$

and

$$\tilde{\vartheta}_t^H = \tilde{\xi}_t^H + \lambda_t \mathcal{E}_{t-} \left( \int_0^{t-} \frac{dH_u - \tilde{\xi}_u^H dS_u}{\mathcal{E}_u} + \int_0^{t-} \frac{\alpha_u \beta_u (\Gamma_u I_u - \beta_u K_u)}{\mathcal{E}_{u-} (\beta_u^2 + \Gamma_u)^2} du \right), \quad (3.2)$$

where  $\mathcal{E}_t := \tilde{Z}_t / \tilde{Z}_0 = \mathcal{E}_t(-\int_0^t \lambda_u dS_u)$  and

$$\tilde{\xi}_t^H := \frac{\beta_t I_t + K_t}{S_{t-} (\beta_t^2 + \Gamma_t)}. \quad (3.3)$$

**Remark 3.2** *As shown in [4],  $\tilde{\xi}^H$  defined in (3.3) represents the LRM strategy for claim  $H$ , that is, for each  $t \in [0, T]$ ,  $\tilde{\xi}_t^H$  gives the number of shares of the risky asset in the LRM strategy at time  $t$ .*

**Remark 3.3** *Benth et al. [5] treated MVH strategies using Malliavin calculus for Lévy processes under the assumption that  $S$  is a martingale. This assumption is very restrictive, and also makes the problem easy.*

**Example 3.4 (Exponential Lévy models)** *As a typical and simple model framework of  $S$ , we consider exponential Lévy models, that is, the case where  $\log(S_t/S_0)$  is a Lévy process represented as*

$$\log(S_t) - \log(S_0) = \mu t + \sigma W_t + \int_{\mathbb{R}_0} z \tilde{N}([0, t], dz)$$

for  $t \in [0, T]$ , where  $\mu \in \mathbb{R}$ ,  $\sigma \geq 0$ . Under Assumption 2.1,  $S$  is a solution to the following stochastic differential equation:

$$dS_t = S_{t-} \left( \mu^S dt + \sigma dW_t + \int_{\mathbb{R}_0} (e^z - 1) \tilde{N}(dt, dz) \right),$$

where  $\mu^S = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}_0} (e^z - 1 - z)v(dz)$ . Supposing Assumption 2.7 additionally, we have, by Theorem 3.1

$$\tilde{\vartheta}_t^H = \tilde{\xi}_t^H + \frac{\mu^S \mathcal{E}_{t-}}{S_{u-}(\sigma^2 + \Gamma)} \left( \int_0^{t-} \frac{dH_u - \tilde{\xi}_u^H dS_u}{\mathcal{E}_u} + \int_0^{t-} \frac{\mu^S \sigma (\Gamma I_u - \sigma K_u)}{\mathcal{E}_{u-}(\sigma^2 + \Gamma)^2} du \right), \quad (3.4)$$

where  $\Gamma := \int_{\mathbb{R}_0} (e^z - 1)^2 v(dz)$ . As seen in [4], the condition

$$\int_{\mathbb{R}_0} \left\{ z^2 + (e^z - 1)^4 \right\} v(dz) < \infty$$

guarantees Assumption 2.7 for options introduced in Subsection 3.2. In addition, [3] introduced a numerical method to compute  $I_t$ ,  $K_t$ ,  $H_t$  and  $\tilde{\xi}_t^H$  for the case where  $H$  is a call option.

### 3.1 Proof of Theorem 3.1

*Step 1:* [1] obtained a similar feedback-form representation to [11] for more general discontinuous semimartingale models. In [1], he defined a new decomposition of  $H$ , which is different from the FS decomposition. Remark that [1] treated, instead of (3.1), the following minimization problem:

$$\min_{\vartheta \in \Theta} \mathbb{E} \left[ (H - G_T(\vartheta))^2 \right]. \quad (3.5)$$

Now, we introduce an outline of the argument in [1].

Recall that Proposition 2.4 holds true under Assumption 2.1. This fact ensures that our setting satisfies Assumption 1 of [1]. Thus, the solution to (3.5) exists, and we denote it by  $\hat{\vartheta}^H \in \Theta$ . We define a new probability measure  $\tilde{\mathbb{Q}}$  as

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}^*} = \frac{\tilde{Z}_T}{\tilde{Z}_0} = \mathcal{E}_T \left( - \int_0^\cdot \lambda_u dS_u \right).$$

As shown in (4.7) of [1],  $H$  admits the following decomposition:

$$H = \mathbb{E}_{\mathbb{P}^*}[H] + G_T(\hat{\eta}^H) + \hat{N}_T^H, \quad (3.6)$$

where  $\hat{\eta}^H \in \Theta$ , and  $\hat{N}^H$  is a  $\mathbb{P}^*$ -martingale with  $\hat{N}_0^H = 0$ . Here  $\hat{N}^H$  is represented as

$$\hat{N}_t^H = \int_0^t \tilde{Z}_{u-} d\hat{L}_u^H + [\tilde{Z}, \hat{L}]_t \quad (3.7)$$

with a square integrable  $\tilde{\mathbb{Q}}$ -martingale  $\hat{L}^H$ . Note that the processes  $\hat{L}^H$  and  $S\hat{L}^H$  both are  $\mathbb{P}^*$ -martingales. Remark that the decomposition (3.6) is neither the Kunita-Watanabe one nor the FS one in our setting. Furthermore, (4.5) in [1] provides that  $\hat{\vartheta}^H$  is given by

$$\hat{\vartheta}_t^H = \hat{\eta}_t^H + \mathbb{E}_{\mathbb{P}^*}[H] \lambda_t \mathcal{E}_{t-} + \hat{L}_{t-}^H \lambda_t \tilde{Z}_{t-}. \quad (3.8)$$

*Step 2:* Replacing  $H$  with the constant 1 in (3.5), we consider the following minimization problem:

$$\min_{\vartheta \in \Theta} \mathbb{E} \left[ (1 - G_T(\vartheta))^2 \right]. \quad (3.9)$$

Letting  $\hat{\vartheta}_t^1 := \lambda_t \mathcal{E}_{t-}$ , we have  $1 - G_T(\hat{\vartheta}^1) = \mathcal{E}_T$ ,  $\mathbb{E}[\mathcal{E}_T \cdot G_T(\vartheta)] = 0$  for any  $\vartheta \in \Theta$ , and  $\mathbb{E}[\mathcal{E}_T] = \mathbb{E}[\mathcal{E}_T^2]$ . Thus, we obtain

$$\begin{aligned} & \mathbb{E}[(1 - G_T(\vartheta))^2] \\ &= \mathbb{E}[(G_T(\vartheta) - G_T(\hat{\vartheta}^1))^2] + 2\mathbb{E}[\mathcal{E}_T(1 - G_T(\vartheta))] - \mathbb{E}[(1 - G_T(\hat{\vartheta}^1))^2] \\ &\geq 2\mathbb{E}[\mathcal{E}_T] - \mathbb{E}[(1 - G_T(\hat{\vartheta}^1))^2] = \mathbb{E}[(1 - G_T(\hat{\vartheta}^1))^2] \end{aligned}$$

for any  $\vartheta \in \Theta$ , which means that  $\hat{\vartheta}^1$  is the solution to (3.9). Hence, Theorem 4.2 of Hou and Karatzas [10] implies that  $\tilde{c}^H = \mathbb{E}_{\mathbb{P}^*}[H]$ , and  $\hat{\vartheta}^{H-c} = \hat{\vartheta}^H - c\hat{\vartheta}^1$  for any  $c \in \mathbb{R}$ , where  $\hat{\vartheta}^{H-c}$  is the solution to the following minimization problem:

$$\min_{\vartheta \in \Theta} \mathbb{E} \left[ (H - c - G_T(\vartheta))^2 \right]$$

for fixed  $c \in \mathbb{R}$ . As a result, we obtain from (3.8)

$$\tilde{\vartheta}_t^H = \hat{\vartheta}^{H-\tilde{c}^H} = \hat{\vartheta}^H - \mathbb{E}_{\mathbb{P}^*}[H]\hat{\vartheta}^1 = \hat{\eta}_t^H + \hat{L}_{t-}^H \lambda_t \tilde{Z}_{t-}. \quad (3.10)$$

*Step 3:* All we have to do is to derive representations of  $\hat{\eta}^H$  and  $\hat{L}^H$ . To this end, we prepare some notation. The Girsanov theorem implies that

$$W_t^{\tilde{\mathbb{Q}}} := W_t^{\mathbb{P}^*} + \int_0^t \lambda_u S_{u-} \beta_u du \quad (3.11)$$

is a one-dimensional Brownian motion under  $\tilde{\mathbb{Q}}$ , and

$$\tilde{N}^{\tilde{\mathbb{Q}}}(dt, dz) := \tilde{N}^{\mathbb{P}^*}(dt, dz) + \lambda_t S_{t-} \gamma_{t,z} \nu_t^{\mathbb{P}^*}(dz) dt \quad (3.12)$$

is the compensated Poisson random measure of  $N$  under  $\tilde{\mathbb{Q}}$ . Denoting

$$\nu_t^{\tilde{\mathbb{Q}}}(dz) = (1 - \lambda_t S_{t-} \gamma_{t,z}) \nu_t^{\mathbb{P}^*}(dz),$$

we have  $\tilde{N}^{\tilde{\mathbb{Q}}}(dt, dz) = N(dt, dz) - \nu_t^{\tilde{\mathbb{Q}}}(dz) dt$  and  $\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{N}^{\tilde{\mathbb{Q}}}(A, B)] = 0$  for any  $A \in \mathcal{B}([0, T])$  and  $B \in \mathcal{B}(\mathbb{R}_0)$ .

*Step 4:* From the view of the martingale representation property under  $\tilde{\mathbb{Q}}$  (Theorem III.4.34 of Jacod and Shiryaev [12]),  $\hat{L}^H$  is described as, for any  $t \in [0, T]$ ,

$$\hat{L}_t^H = \int_0^t \hat{l}_u^W dW_u^{\tilde{\mathbb{Q}}} + \int_0^t \int_{\mathbb{R}_0} \hat{l}_{u,z}^N \tilde{N}^{\tilde{\mathbb{Q}}}(du, dz) \quad (3.13)$$

for some predictable processes  $\hat{l}^W$  and  $\hat{l}^N$ . Since the product process  $S\hat{L}^H$  is a martingale under  $\mathbb{P}^*$ , we get

$$\beta_t \hat{l}_t^W + \int_{\mathbb{R}_0} \gamma_{t,z} \hat{l}_{t,z}^N \nu_t^{\mathbb{P}^*}(dz) = 0 \quad (3.14)$$

for any  $t \in [0, T]$ . (3.7) provides

$$\hat{N}_T^H = \int_0^T \tilde{Z}_u - \hat{l}_u^W dW_u^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \tilde{Z}_u - \hat{l}_{u,z}^N (1 - \lambda_u S_u - \gamma_{u,z}) \tilde{N}^{\mathbb{P}^*}(du, dz), \quad (3.15)$$

since the condition (3.14) implies that

$$\begin{aligned} [\tilde{Z}, \hat{L}^H]_T &= - \int_0^T \lambda_u \tilde{Z}_u - d[S, \hat{L}^H]_u \\ &= - \int_0^T \lambda_u \tilde{Z}_u - S_u - \left( \beta_u \hat{l}_u^W du + \int_{\mathbb{R}_0} \gamma_{u,z} \hat{l}_{u,z}^N N(du, dz) \right) \\ &= - \int_0^T \lambda_u \tilde{Z}_u - S_u - \int_{\mathbb{R}_0} \gamma_{u,z} \hat{l}_{u,z}^N \tilde{N}^{\mathbb{P}^*}(du, dz). \end{aligned}$$

Then, (3.6) together with (3.15) provides

$$\begin{aligned} H &= \mathbb{E}_{\mathbb{P}^*}[H] + \int_0^T (\hat{\eta}_u^H S_u - \beta_u + \tilde{Z}_u - \hat{l}_u^W) dW_u^{\mathbb{P}^*} \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \left( \hat{\eta}_u^H S_u - \gamma_{u,z} + \tilde{Z}_u - \hat{l}_{u,z}^N (1 - \lambda_u S_u - \gamma_{u,z}) \right) \tilde{N}^{\mathbb{P}^*}(du, dz). \end{aligned} \quad (3.16)$$

Comparing (3.16) with (2.5), we obtain, for any  $t \in [0, T]$ ,

$$\begin{cases} \hat{\eta}_t^H S_{t-} - \beta_t + \tilde{Z}_{t-} - \hat{l}_t^W = I_t, \\ \hat{\eta}_t^H S_{t-} - \gamma_{t,z} + \tilde{Z}_{t-} - \hat{l}_{t,z}^N (1 - \lambda_t S_{t-} - \gamma_{t,z}) = J_{t,z}, \end{cases} \quad (3.17)$$

which yields

$$\hat{l}_t^W = \frac{1}{\tilde{Z}_{t-}} (I_t - \hat{\eta}_t^H S_{t-} - \beta_t),$$

and

$$\hat{l}_{t,z}^N (1 - \lambda_t S_{t-} - \gamma_{t,z}) = \frac{1}{\tilde{Z}_{t-}} (J_{t,z} - \hat{\eta}_t^H S_{t-} - \gamma_{t,z}).$$

Solving the simultaneous equation (3.17) on  $\hat{\eta}^H$  by using (2.4) and (3.14), we have

$$\hat{\eta}_t^H = \frac{\beta_t I_t + K_t}{S_{t-} (\beta_t^2 + \Gamma_t)}.$$

(3.3) implies that  $\hat{\eta}^H$  coincides with  $\tilde{\zeta}^H$ , which is the LRM strategy for  $H$ .

Consequently, we get

$$\hat{l}_t^W = \frac{1}{\tilde{Z}_{t-}} \frac{\Gamma_t I_t - \beta_t K_t}{\beta_t^2 + \Gamma_t}$$

as well as

$$\widehat{l}_{t,z}^N(1 - \lambda_t S_{t-} \gamma_{t,z}) = \frac{1}{\widetilde{Z}_{t-}} \left( J_{t,z} - \frac{\beta_t I_t + K_t}{\beta_t^2 + \Gamma_t} \gamma_{t,z} \right).$$

Noting that

$$\frac{\Gamma_t I_t - \beta_t K_t}{\beta_t^2 + \Gamma_t} \beta_t + \int_{\mathbb{R}_0} \left( J_{t,z} - \frac{\beta_t I_t + K_t}{\beta_t^2 + \Gamma_t} \gamma_{t,z} \right) \gamma_{t,z} \nu(dz) = 0,$$

we obtain from (3.11), (3.12) and (3.13) that

$$d\widehat{L}_t^H = \frac{1}{\widetilde{Z}_{t-}} \frac{\Gamma_t I_t - \beta_t K_t}{\beta_t^2 + \Gamma_t} dW_t^{\mathbb{P}^*} + \frac{1}{\widetilde{Z}_{t-}} \int_{\mathbb{R}_0} \frac{J_{t,z} - \frac{\beta_t I_t + K_t}{\beta_t^2 + \Gamma_t} \gamma_{t,z}}{1 - \lambda_t S_{t-} \gamma_{t,z}} \widetilde{N}^{\mathbb{P}^*}(dt, dz). \quad (3.18)$$

*Step 5:* From the view of (3.10) together with (3.18), we calculate  $\widetilde{\vartheta}^H$  as follows:

$$\begin{aligned} \widetilde{\vartheta}_t^H &= \widehat{\eta}_t^H + \widehat{L}_{t-}^H \lambda_t \widetilde{Z}_{t-} \\ &= \widetilde{\xi}_t^H + \lambda_t \widetilde{Z}_{t-} \left( \int_0^{t-} \frac{1}{\widetilde{Z}_{u-}} \frac{\Gamma_u I_u - \beta_u K_u}{\beta_u^2 + \Gamma_u} dW_u^{\mathbb{P}^*} \right. \\ &\quad \left. + \int_0^{t-} \int_{\mathbb{R}_0} \frac{1}{\widetilde{Z}_{u-}} \frac{1}{1 - \lambda_u S_{u-} \gamma_{u,z}} \left( J_{u,z} - \frac{\beta_u I_u + K_u}{\beta_u^2 + \Gamma_u} \gamma_{u,z} \right) \widetilde{N}^{\mathbb{P}^*}(du, dz) \right) \\ &=: \widetilde{\xi}_t^H + \lambda_t \widetilde{Z}_{t-} (\Xi_{t-}^W + \Xi_{t-}^N). \end{aligned}$$

We have

$$\begin{aligned} \Xi_{t-}^W &= \int_0^{t-} \frac{1}{\widetilde{Z}_{u-}} \frac{\Gamma_u I_u - \beta_u K_u}{\beta_u^2 + \Gamma_u} dW_u^{\mathbb{P}^*} \\ &= \int_0^{t-} \frac{I_u}{\widetilde{Z}_{u-}} dW_u^{\mathbb{P}^*} - \int_0^{t-} \frac{\beta_u I_u + K_u}{\beta_u^2 + \Gamma_u} \frac{\beta_u}{\widetilde{Z}_{u-}} dW_u^{\mathbb{P}^*} \\ &= \int_0^{t-} \frac{I_u}{\widetilde{Z}_u} dW_u^{\mathbb{P}^*} - \int_0^{t-} \frac{\widetilde{\xi}_u^H}{\widetilde{Z}_u} S_{u-} \beta_u dW_u^{\mathbb{P}^*}, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \Xi_{t-}^N &= \int_0^{t-} \int_{\mathbb{R}_0} \frac{1}{\widetilde{Z}_{u-}} \frac{1}{1 - \lambda_u S_{u-} \gamma_{u,z}} \left( J_{u,z} - \frac{\beta_u I_u + K_u}{\beta_u^2 + \Gamma_u} \gamma_{u,z} \right) \widetilde{N}^{\mathbb{P}^*}(du, dz) \\ &= \int_0^{t-} \int_{\mathbb{R}_0} \frac{1}{\widetilde{Z}_{u-}} \frac{1}{1 - \lambda_u S_{u-} \gamma_{u,z}} (J_{u,z} - \widetilde{\xi}_u^H S_{u-} \gamma_{u,z}) \widetilde{N}^{\mathbb{P}^*}(du, dz) \\ &= \int_0^{t-} \int_{\mathbb{R}_0} (J_{u,z} - \widetilde{\xi}_u^H S_{u-} \gamma_{u,z}) \left( \frac{N(du, dz)}{\widetilde{Z}_u} - \frac{\nu_u^{\mathbb{P}^*}(dz) du}{\widetilde{Z}_{u-} (1 - \lambda_u S_{u-} \gamma_{u,z})} \right) \\ &= \int_0^{t-} \int_{\mathbb{R}_0} (J_{u,z} - \widetilde{\xi}_u^H S_{u-} \gamma_{u,z}) \left( \frac{\widetilde{N}^{\mathbb{P}^*}(du, dz)}{\widetilde{Z}_u} - \frac{\lambda_u S_{u-} \gamma_{u,z} \nu(dz) du}{\widetilde{Z}_{u-}} \right) \end{aligned}$$

$$= \int_0^{t-} \int_{\mathbb{R}_0} (J_{u,z} - \tilde{\xi}_u^H S_u - \gamma_{u,z}) \frac{\tilde{N}^{\mathbb{P}^*}(du, dz)}{\tilde{Z}_u} + \int_0^{t-} \frac{\alpha_u \beta_u (\Gamma_u I_u - \beta_u K_u)}{\tilde{Z}_u - (\beta_u^2 + \Gamma_u)^2} du. \quad (3.20)$$

Thus, from (3.19) and (3.20), we obtain

$$\Xi_{t-}^W + \Xi_{t-}^N = \int_0^{t-} \frac{dH_u - \tilde{\xi}_u^H dS_u}{\tilde{Z}_u} + \int_0^{t-} \frac{\alpha_u \beta_u (\Gamma_u I_u - \beta_u K_u)}{\tilde{Z}_u - (\beta_u^2 + \Gamma_u)^2} du.$$

This completes the proof of Theorem 3.1.  $\square$

### 3.2 Examples

**Example 3.5 (Call and Asian options)** We consider two representative options as contingent claims to hedge: call options  $(S_T - K)^+$  and Asian options  $(\frac{1}{T} \int_0^T S_u du - K)^+$  for  $K > 0$ . In order to obtain explicit representations of  $\tilde{\vartheta}^H$  for such options, we have only to show expressions of  $I_t$  and  $J_{t,z}$  from the view of (3.2). In addition to Assumption 2.1, we assume the following condition:

$$\int_{\mathbb{R}_0} \{\gamma_{t,z}^4 + |\log(1 + \gamma_{t,z})|^2\} \nu(dz) < C \text{ for some } C > 0,$$

which ensures Assumption 2.7 as seen in Sections 4 and 5 of [4]. For  $K > 0$  and  $t \in [0, T]$ , we have that

$$\begin{cases} I_t &= \beta_t \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T > K\}} S_T | \mathcal{F}_{t-}], \\ J_{t,z} &= \mathbb{E}_{\mathbb{P}^*}[(S_T(1 + \gamma_{t,z}) - K)^+ - (S_T - K)^+ | \mathcal{F}_{t-}] \end{cases}$$

for call options  $(S_T - K)^+$ , and

$$\begin{cases} I_t &= \beta_t \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{V_0 > K\}} V_t | \mathcal{F}_{t-}], \\ J_{t,z} &= \mathbb{E}_{\mathbb{P}^*}[(V_0 + \gamma_{t,z} V_t - K)^+ - (V_0 - K)^+ | \mathcal{F}_{t-}], \end{cases}$$

where  $V_t = \frac{1}{T} \int_t^T S_u du$ , for Asian options  $(\frac{1}{T} \int_0^T S_u du - K)^+$ .

**Example 3.6 (Lookback options)** The payoff of a call option is given as a function of  $S_T$ . On the other hand, that of an Asian option depends on the whole trajectory of  $S$ . Such options are said to be path-dependent. Various types of path-dependent options have been traded actively in recent years, but they are excluded in [11]. As one more typical example of path-dependent options covered by Theorem 3.1, we deal with a lookback option, whose payoff depends on the running maximum of  $S$ . In particular, we consider the case of  $H = (M^S - K)^+$ , where  $M^S := \sup_{t \in [0, T]} S_t$  and  $K > 0$ . For an exponential Lévy model introduced in Example 3.4, Section 6 of [4] implies that, under Assumption 2.1 and the condition  $\int_{\mathbb{R}_0} \{z^2 + (e^z - 1)^4\} \nu(dz) < \infty$ ,

$$\begin{cases} I_t &= \sigma \mathbb{E}_{\mathbb{P}^*}[M^S \mathbf{1}_{\{\log(M^S) > \log(K/S_0)\}} \mathbf{1}_{\{\tau \geq t\}} | \mathcal{F}_{t-}], \\ J_{t,z} &= \mathbb{E}_{\mathbb{P}^*} \left[ \left( \sup_{u \in [0, T]} \left( S_u e^{z \mathbf{1}_{\{t \leq u\}}} \right) - K \right)^+ - (M^S - K)^+ \middle| \mathcal{F}_{t-} \right], \end{cases}$$

where  $\tau := \inf\{t \in [0, T] | S_t \vee S_{t-} = M^S\}$ .

## 4 Numerical analysis

We shall develop a simple numerical scheme of the values of MVH strategy  $\tilde{\vartheta}_t^H$  at non-zero time  $t \in (0, T]$  for exponential Lévy models, and illustrate in subsection 4.1 some numerical results. To our best knowledge, no numerical methods for the values of  $\tilde{\vartheta}_t^H$  at non-zero time  $t \in (0, T]$  have been developed. There are two possible reasons for it. First, little has been known about explicit closed-form expressions of  $\tilde{\vartheta}^H$ . As said in Introduction, feedback-form representations are not suitable for the development of numerical methods. For more details on this matter, see Remark 4.1 below. Second, the value of  $\tilde{\vartheta}_t^H$  for  $t \in (0, T]$  is depending on not only  $S_{t-}$  but also the whole trajectory of  $S$  from 0 to  $t-$ . However, it is impossible to observe the trajectory of  $S$  continuously from a practical point of view. Accordingly, we compute  $\tilde{\vartheta}_t^H$  approximately using discrete observational data  $S_{t_0}, S_{t_1}, \dots, S_{t_n}$ , where  $n \geq 1$  and  $t_k = \frac{k}{n+1}t$  for  $k = 0, \dots, n$ . Remark that we divide the time interval  $[0, t]$  equally for simplicity.

We consider in this section an exponential Lévy model introduced in Example 3.4, and denote

$$\begin{cases} H_{t_k} &= \mathbb{E}_{\mathbb{P}^*}[H|S_{t_k}], \\ I_{t_k} &= \mathbb{E}_{\mathbb{P}^*}[D_{t_k,0}H|S_{t_{k-1}}], \\ K_{t_k} &= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[zD_{t_k,z}H|S_{t_{k-1}}](e^z - 1)\nu(dz), \\ \tilde{\zeta}_{t_k}^H &= \frac{\sigma I_{t_k} + K_{t_k}}{S_{t_{k-1}}(\sigma^2 + \Gamma)} \end{cases}$$

for  $k = 1, \dots, n+1$ , where  $t_{n+1} = t$ . Remark that all  $H_{t_k}$ ,  $I_{t_k}$  and  $K_{t_k}$  are computable by the FFT-based approach developed in [3].  $\tilde{\vartheta}_t^H$  is then approximated from the view of (3.4) as

$$\tilde{\vartheta}_t^H \approx \tilde{\zeta}_t^H + \frac{\mu^S \mathcal{E}_{t_n}}{S_{t_n}(\sigma^2 + \Gamma)} \left( \sum_{k=1}^n \frac{\Delta H_{t_k} - \tilde{\zeta}_{t_k}^H \Delta S_{t_k}}{\mathcal{E}_{t_k}} + \sum_{k=1}^n \frac{\mu^S \sigma (\Gamma I_{t_k} - \sigma K_{t_k})}{\mathcal{E}_{t_{k-1}}(\sigma^2 + \Gamma)^2} \Delta t_k \right), \quad (4.1)$$

where  $\Delta X_{t_k} := X_{t_k} - X_{t_{k-1}}$  for any sequence  $\{X_{t_k}\}$ , and  $H_{t_0} := \mathbb{E}_{\mathbb{P}^*}[H]$ . Moreover, we approximate  $\mathcal{E}_{t_k}$  using a recursive calculation as  $\mathcal{E}_0 = 1$  and

$$\mathcal{E}_{t_{k+1}} = \mathcal{E}_{t_k} \left\{ 1 - \frac{\mu^S}{\sigma^2 + \Gamma} \frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right\} = \prod_{l=1}^k \left\{ 1 - \frac{\mu^S}{\sigma^2 + \Gamma} \frac{\Delta S_{t_{l+1}}}{S_{t_l}} \right\} \quad (4.2)$$

for  $k = 1, \dots, n$ , which is a discretization of a stochastic exponential.

**Remark 4.1** An approximation of  $\tilde{\vartheta}^H$  using a feedback-form expression is basically given as a discretization of a general stochastic exponential, defined as a solution  $Y$  to



the following type of stochastic differential equation:

$$Y_t = U_t + \int_0^t Y_u - dV_u,$$

where  $U$  and  $V$  are semimartingales. Theorem V.52 of [16] implies that, if  $V$  is continuous, then  $Y$  is given as

$$Y_t = \mathcal{E}_t(V) \left\{ U_0 + \int_{0+}^t \mathcal{E}_u(V)^{-1} (dU_u - d[U, V]_u) \right\},$$

which is much more complicated than ordinary stochastic exponentials. As a result, a recursive calculation for a discretization of  $Y$  is involved in contrast to (4.2), which means that feedback-form expressions are not appropriate to develop an approximation method for  $\tilde{\vartheta}^H$ .

**Remark 4.2** It is almost impossible to develop a similar approximation method to (4.1) using the closed-form expression obtained by [11], since their expression, which is given as a general stochastic exponential, includes a stochastic integral with respect to the quadratic variation of  $S$ , which we cannot observe directly.

## 4.1 Numerical results

We focus on the case where the process  $\log(S/S_0)(=: L^S)$ , is given as a variance Gamma process, and  $H$  is a call option  $H = (S_T - K)^+$  with  $K > 0$ . Note that a variance Gamma process is defined as a time-changed Brownian motion subject to a gamma subordinator. In summary,  $L^S$  is represented as

$$L_t^S = mG_t + \delta B_{G_t} \quad \text{for } t \in [0, T],$$

where  $\delta > 0$ ,  $m \in \mathbb{R}$ ,  $B$  is a one-dimensional standard Brownian motion, and  $G_t$  is a gamma process with parameters  $(1/\kappa, 1/\kappa)$  for  $\kappa > 0$ . Its Lévy measure is then given as

$$\nu(dz) = C \left( \mathbf{1}_{\{z < 0\}} e^{-G|z|} + \mathbf{1}_{\{z > 0\}} e^{-M|z|} \right) \frac{dz}{|z|},$$

where

$$C := \frac{1}{\kappa} > 0, \quad G := \frac{1}{\delta^2} \sqrt{m^2 + \frac{2\delta^2}{\kappa}} + \frac{m}{\delta^2} > 0, \quad M := \frac{1}{\delta^2} \sqrt{m^2 + \frac{2\delta^2}{\kappa}} - \frac{m}{\delta^2} > 0.$$

Note that  $L^S$  has no Brownian component, that is,  $\sigma$  in Example 3.4 is given by 0. As a result, the approximation (4.1) for  $\tilde{\vartheta}_t^H$  is simplified as

$$\tilde{\vartheta}_t^H \approx \tilde{\zeta}_t^H + \frac{\mu^S \mathcal{E}_{t_n}}{S_{t_n} \Gamma} \sum_{k=1}^n \frac{\Delta H_{t_k} - \tilde{\zeta}_{t_k}^H \Delta S_{t_k}}{\mathcal{E}_{t_k}},$$

where

$$\mu^S = \int_{\mathbb{R}_0} (e^z - 1)v(dz), \quad \tilde{\xi}_{t_k}^H = \frac{K_{t_k}}{S_{t_{k-1}}\Gamma} \quad \text{and} \quad \mathcal{E}_{t_{k+1}} = \mathcal{E}_{t_k} \left\{ 1 - \frac{\mu^S \Delta S_{t_{k+1}}}{S_{t_k}\Gamma} \right\}.$$

Recall that  $H_{t_k}$  and  $K_{t_k}$  are computed with the FFT-based scheme developed in [3].

We consider European call options on the S&P 500 Index matured on 19 May 2017, and set the initial date of our hedging to 20 May 2016. We fix  $T$  to 1. There are 251 business days on and after 20 May 2016 until and including 19 May 2017. Thus, for example, 20 May 2016 and 23 May 2016 are corresponding to time 0 and  $1/250$ , respectively, since 20 May 2016 is Friday. We compute the values of MVH strategies  $\tilde{\theta}^H$  on 10 November 2016. Since 10 November 2016 is the 121st business day after 20 May 2016, letting  $t = 121/250$ , we compute the values of  $\tilde{\theta}_t^H$ . Remark that  $\tilde{\theta}_t^H$  is constructed on 9 November 2016. Thus, it is computed using 121 dairy closing prices of S&P 500 index on and after 20 May 2016 until and including 9 November 2016 as discrete observational data. As contingent claims to hedge, we consider call options with strike price 1500, 1550,  $\dots$ , 2500. In addition, we set model parameters as  $C = 6.7910$ ,  $G = 30.1807$ , and  $M = 33.1507$ , which are calibrated by the data set of European call options on the S&P 500 Index at 20 April 2016. Note that the above parameter set was used in Arai and Imai [2], and satisfies Assumptions 2.1 and 2.7. Figure 1 shows the values of  $\tilde{\theta}_t^H$ . The computation time to obtain the 21 values of  $\tilde{\theta}_t^H$  on Figure 1 is 28.85 s, which indicates that we achieve fast computation using an FFT-based method, nevertheless computation for MVH strategies is time-consuming in general. In addition, we compute the values of  $\tilde{\theta}_t^H - \tilde{\xi}_t^H$ , which is the difference between the values of the MVH and LRM strategies. Figure 2 shows that the differences are very small, more precisely, the absolute values of  $\tilde{\theta}_t^H - \tilde{\xi}_t^H$  are no more than 0.0025. Note that our numerical experiments are carried out using MATLAB (9.0.0.341360 R2016a) on an Intel Core i7 3.4 GHz CPU with 16 GB 1333 MHz DDR3 memory.

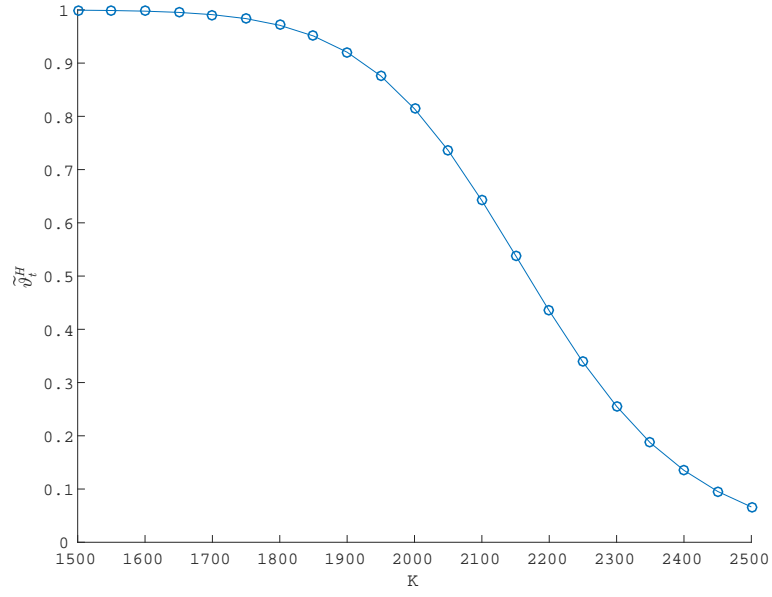


Figure 1: Values of  $\tilde{\theta}_t^H$  of call options at  $t = 121/250$  vs. strike price  $K$  from 1500 to 2500 at steps of 50.

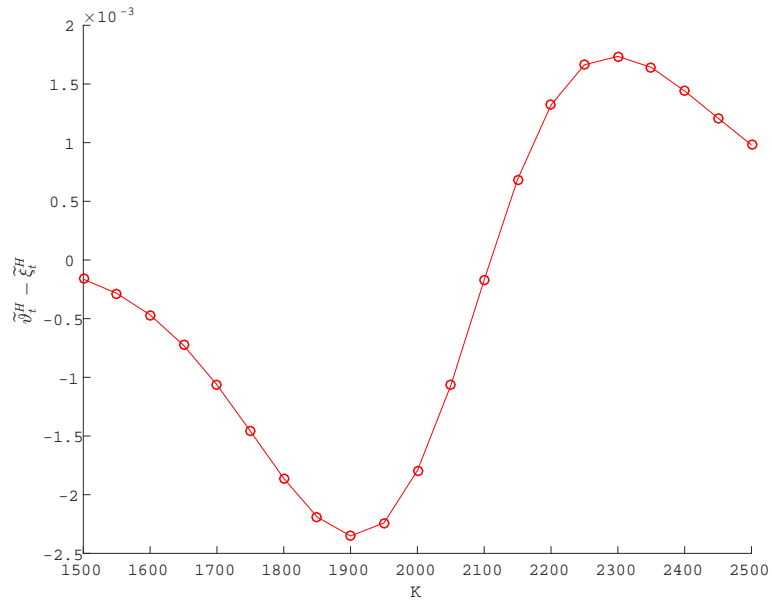


Figure 2: Values of  $\tilde{\theta}_t^H - \tilde{\zeta}_t^H$  vs. strike price  $K$ .

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